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Minimum-Effort Interception of Multiple Targets

Joseph Z. Ben-Asher*
Tel-Aviv University, Tel-Aviv 69978, Israel

I. Introduction

A WELL-KNOWN fact about the widely used proportional navigation law is that it is the solution to the minimum-effort interception problem where a zero miss distance is required as the terminal boundary condition.[1] This result provides the motivation for studying this minimum-effort problem with more than one target. Norbutas[2] formulated a set of n two-point boundary-value problems coupled through their boundary conditions. For the general case, he suggested some numerical solutions for open-loop control. He succeeded in finding an analytical solution in feedback form (i.e., closed-loop) only for the two-target case. The main contribution of this work is a generalization of the closed-loop analytical solution to the n-target case. Both proportional navigation (i.e., the solution to the one-target case) and Norbutas' two-target optimal solution are shown to be special cases of the general feedback solution.

II. Problem Formulation

Assume n stationary targets located in the plane in the neighborhood of a nominal line of sight (LOS) at fixed normal displacements $x_1, x_2, \ldots, x_n$. Without any loss of generality, we let $x_n = 0$. The nominal times at which the interceptor approaches these targets (i.e., passes through their projections on the LOS) are $t_1, t_2, \ldots, t_n$.

Under the assumption that the speed V is constant, the equations of motion of the interceptor (in the plane) are the following:

$$\begin{aligned} r'(t) &= V\cos[\alpha(t)] \\ x'(t) &= V\sin[\alpha(t)] \end{aligned} \tag{1}$$

where t is the elapsed time, $r(t)$ the projection of the trajectory on the nominal LOS, $x(t)$ the normal displacement, $\alpha(t)$ the flight-path angle, and $()'$ signifies differentiation with respect to time. If $\alpha(t)$ is small, we get

$$\begin{aligned} r'(t) &\approx V \\ x'(t) &= V\alpha(t) \equiv u(t) \end{aligned} \tag{2}$$

Assuming that the interceptor has direct control over its normal acceleration $a(t)$, we obtain the following state equations:

$$\begin{aligned} x'(t) &= u(t) \\ u'(t) &= a(t) \end{aligned} \tag{3}$$

The optimal control problem is to minimize the control effort

$$J = \frac{1}{2}\int_0^{t_n} a^2(t)\, dt \tag{4}$$

subject to the following conditions:

$$\begin{aligned} x(t_1) &= x_1 \\ x(t_2) &= x_2 \\ &\vdots \\ x(t_n) &= x_n \end{aligned} \tag{5}$$

and the initial conditions

$$\begin{aligned} x(t_0) &= x_0 \\ u(t_0) &= u_0 \end{aligned} \tag{6}$$

III. Problem Analysis

Let $H(x, u, a, p_x, p_u)$ be the usual Hamiltonian defined by

$$H(x, u, a, p_x, p_u) = \tfrac{1}{2}a^2 + p_x u + p_u a \tag{7}$$

The Euler-Lagrange's equations are

$$\begin{aligned} p_x'(t) &= 0 \\ p_u'(t) &= -p_x(t) \end{aligned} \tag{8}$$

By the minimum principle, $a(t)$ should minimize H, thus

$$a(t) = -p_u(t) \tag{9}$$

The transversality conditions are the following[1,2]:

$$\begin{aligned} p_x(t_i^+) &= p_x(t_i^-) + \alpha_i \qquad i = 1, \ldots, n-1 \\ p_u(t_i^+) &= p_u(t_i^-) \qquad i = 1, \ldots, n-1 \end{aligned} \tag{10}$$

and at the terminal time

$$p_u(t_n) = 0 \tag{11}$$

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*Adjunct Professor, Department of Electronic Systems. Member AIAA.

Table 1 Feedback gains

k_x	$\dfrac{-3(6h_1h_2+3h_2^2+6h_1h_3+4h_2h_3)}{h_1^2(3h_1h_2+3h_2^2+3h_1h_3+4h_2h_3)}$
k_u	$\dfrac{-3(4h_1h_2+3h_2^2+4h_1h_3+4h_2h_3)}{h_1(3h_1h_2+3h_2^2+3h_1h_3+4h_2h_3)}$
k_{x1}	$\dfrac{3(h_1+h_2)(3h_1h_2+3h_2^2+2h_1h_3+4h_2h_3)}{h_2h_1^2(3h_1h_2+3h_2^2+3h_1h_3+4h_2h_3)}$
k_{x2}	$\dfrac{-3(h_2+h_3)(h_2+2h_3)}{h_2h_3(3h_1h_2+3h_2^2+3h_1h_3+4h_2h_3)}$

The formulation of the n two-point boundary-value problems is now complete, and, in principle, the problem can be solved numerically by standard methods. However, we make the following observations:

1) p_u and, therefore, the optimal control $a(t)$ are continuous (as opposed to the target-by-target interception where the control is only piecewise continuous and jumps at each intermediate target).

2) p_x is piecewise constant and, therefore, $a(t)$ is piecewise linear.

3) $a(t)$ vanishes at the final time.

IV. Method of Solution

For the single-target case ($n = 1$), the proportional navigation law exists as the solution and takes the form

$$a(t) = 3[x(t)/(t_n - t)^2 + u(t)/(t_n - t)] \tag{12}$$

As indicated, Norbutas[2] formulated the solution for the two-target case ($n = 2$) in closed form as here. For $t \le t_1$,

$$\begin{aligned} a(t) = {} & \{6(3h_1 + 2h_2)/[h_1^2(3h_1 + 4h_2)]\}x(t) \\ & - \{12(h_1 + h_2)/[h_1(3h_1 + 4h_2)]\}u(t) \\ & + \{6(h_1 + h_2)(h_1 + 2h_2)/[h_2h_1^2(3h_1 + 4h_2)]\}x_1 \end{aligned} \tag{13}$$

where $h_1 = t_1 - t$ and $h_2 = t_2 - t_1$. For $t \ge t_1$, Eq. (10) holds (with $n = 2$). By the use of the foregoing observations, the optimization problem can be transformed into one of cubic spline interpolation. Our aim is to interpolate the points (t_0, x_0), $(t_1, x_1), \ldots, (t_n, x_n)$ with cubic polynomials (integrating twice the piecewise linear control), while requiring the continuity of the first and second derivatives, and satisfying a given first derivative u_0 at t_0, and a zero second derivative at t_n. This is a standard interpolation problem.[3] Let $A_0, A_1, \ldots, A_n$ denote the value of $a(t)$ at $t_0, t_1, \ldots, t_n$ and define $P_i''(t)$ to be a linear interpolation between A_{i-1} and A_i on the interval $[t_{i-1}, t_i]$. Thus,

$$a(t) \equiv P_i''(t) = (t_i - t)A_{i-1}/h_i + (t - t_{i-1})A_i/h_i \tag{14}$$

where $h_i = t_i - t_{i-1}$. In addition, we obtain the following requirements for the first and second integrals such that:

$$\begin{aligned} & P_i'(t_i) = P_{i+1}'(t_i), \qquad i = 1, 2, \ldots, n-1 \\ & P_i(t_{i-1}) = x_{i-1}, \qquad P_i(t_i) = x_i, \qquad i = 1, 2, \ldots, n \end{aligned} \tag{15}$$

Moreover, we require that $P_1'(t_0) = u_0$ and $P_n''(t_n) = 0$. From these requirements, one obtains the following relation (see Ref. 3 for details):

$$H\bar{A} = 6W(x_0, u_0, x_1, x_2, \ldots, x_n) \tag{16}$$

where

$$H = \begin{bmatrix} 2h_1 & h_1 & 0 & \cdot & \cdot & \cdot & 0 \\ h_1 & 2(h_1+h_2) & h_2 & & & & \cdot \\ 0 & & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & 0 \\ \cdot & & & \cdot & \cdot & & h_{n-1} \\ 0 & \cdot & \cdot & \cdot\ 0 & h_{n-1} & & 2(h_{n-1}+h_n) \end{bmatrix} \tag{17}$$

$$\bar{A}^T = [A_0, A_1, A_2, \ldots, A_{n-1}] \tag{18}$$

and

$$W(x_0, u_0, x_1, \ldots, x_n) = \begin{bmatrix} (x_1 - x_0)/h_1 - u_0 \\ (x_2 - x_1)/h_2 - (x_1 - x_0)/h_1 \\ \cdot \\ \cdot \\ (x_n - x_{n-1})/h_n - (x_{n-1} - x_{n-2})/h_{n-1} \end{bmatrix} \tag{19}$$

The required acceleration at t_0 can be solved for as follows:

$$A_0 = 6[1, 0, \ldots, 0]\, H^{-1}\, W(x_0, u_0, x_1, \ldots, x_n) \tag{20}$$

Omitting the subscript 0 and noting that H is dependent on t via $h_1(t) = t_1 - t$, we obtain the feedback law

$$a(t) = 6[1, 0, \ldots, 0]\, H^{-1}(t)\, W[x(t), u(t), x_1, \ldots, x_n] \tag{21}$$

Notice that the proportional navigation law is obtained by letting $n = 1$ in the preceding expression, i.e.,

$$a(t) = 6(1)(2h_1)^{-1}\{[x_1 - x(t)]/h_1 - u(t)\} \tag{22}$$

Similarly, by letting $n = 2$, one obtains

$$a(t) = 6[1, 0]\begin{bmatrix} 2h_1 & h_1 \\ h_1 & 2(h_1 + h_2) \end{bmatrix}^{-1} \begin{bmatrix} [x_1 - x(t)]/h_1 - u(t) \\ (x_2 - x_1)/h_2 - [x_1 - x(t)]/h_1 \end{bmatrix} \tag{23}$$

which is (for $x_2 = 0$) in agreement with Eq. (13). Employing a symbolic code, one can obtain expressions similar to Eqs. (12) and (13) for higher values of n. For instance, by letting $n = 3$, we arrive at the following expression by the use of *Mathematica*[4]:

$$a(t) = k_x x(t) + k_u u(t) + k_{x1} x_1 + k_{x2} x_2 \tag{24}$$

where the coefficients are given in Table 1. Note that, in this case, Eq. (24) holds for $t \le t_1$, where for $t_1 \le t \le t_2$ we use Eq. (13), and for $t_2 \le t$ we apply the proportional navigation law (12).

V. Conclusions

The minimum-effort multiple-target interception problem has been solved by identifying it as a cubic-spline interpolation problem. The closed-loop control involves, in general, an on-line inversion of an $n \times n$ matrix. If, however, the number of targets is known, the inversion can be done off-line by present-day symbolic codes to obtain a simple linear feedback law.

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Gain Scheduling Simplification by Simultaneous Control Design

F. M. Al-Sunni*
King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
and
F. L. Lewis†
University of Texas at Arlington, Fort Worth, Texas 76118

I. Introduction

THE robust control problem can be stated as follows: Design a single controller $C(s)$ that achieves prescribed performance over a region of operating conditions of the plant $G(s)$. This performance can be achieved by designing a controller for a representative plant $G_0(s)$ and maximizing the perturbation, around $G_0(s)$, that can be tolerated by the controller $C(s)$. The goal of the maximization is to cover the entire region of operation. This problem has been addressed in Refs. 1-3. Another version of this problem is to look at a "discrete" representation of the region of operation. In this approach, a finite number, say, M, of representative plants are chosen. A controller $C(s)$ is then designed to control all of the plants.[4-12] We will refer to this problem as the *simultaneous control problem.*

One of the motivations behind the simultaneous control problem is to control a nonlinear system, represented by linear models at different operating points, using one controller. In this case, each linear model represents an operating point. The parameter variations of the system model form a low-frequency upper bound on the singular values of the loop-gain transfer function. Some of the robust controller design methods, LQG/LTR, for example, have no mechanism for dealing with this upper bound. On the other hand, simultaneous control is a natural way of approaching this issue. The simultaneous control technique can make gain scheduling easier to implement by reducing the number of operating points to be scheduled. This is accomplished by grouping the total number of prespecified operating points into classes. A different controller is then designed for each of these classes.

Another use for simultaneous control is in the design of controllers that are robust against sensor or actuator failures. When an actuator or a sensor fails during operation, the system characteristics change, effectively generating a new system. Simultaneous control design can be used to design a single controller that gives good performance for both the original system and the new system generated by the failure.

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*Assistant Professor, Department of Systems Engineering.

†Moncrief-O'Donnell Professor, Automation and Robotics Research Institute.

The simultaneous control problem has attracted many researchers over the last few years, but most of their work has dealt with the problem of simultaneous stabilization without meeting any specified performance objectives. Some of the results for single-input/single-output (SISO) transfer functions were reported in Refs. 4–8. The multi-input/output case was addressed in Ref. 9. In that work, a dynamical output-feedback controller is designed to achieve the stabilization of more than one system. A nonlinear feedback controller for SISO systems was used in Ref. 11. A state-feedback controller is proposed in Ref. 12. The design in Ref. 12 is a two-stage design. In one stage a state-feedback gain K is designed to make all of the systems minimum-phase and the pole-zero excess of each system equal to 1. Once this is achieved, a constant output-feedback controller can be designed to stabilize all of the "new" systems.

The only work that deals with the problem of simultaneous attainment of performance objectives is that of Looze.[10] The objective there is to find a state-feedback controller K that minimizes an objective function composed of the sum of M standard LQR cost functions. Each one of these cost functions penalizes the states and controls of one of the M systems. One problem with the work in Ref. 10 is the need for finding an initial stabilizing controller. A method for generating such a controller was given, but it requires a lot of CPU time.

In this paper we address the problem of designing a state-variable feedback controller that stabilizes a finite collection of systems. Our technique has extra design parameters that allow, in addition to stabilization, the achievement of some performance specifications. An initial stabilizing gain is not needed for our technique. We design a controller K_i for each of the M systems S_i and give necessary conditions for the sum of K_i to be a stabilizing controller for all of the systems S_i, $i=1,\ldots,M$.

The main result is presented in Sec. II, followed by a gain-scheduling design example. In the design example we consider the problem of stabilizing the longitudinal short-period approximation of the McDonnell-Douglas F4-E fighter aircraft for different operating points.

II. Simultaneous Control Results and Gain-Scheduling Example

We consider a collection of linear time-invariant systems described by

$$\dot{x}_i = A_i x_i + B_i u_i, \qquad i = 1, 2, \ldots, M \tag{1}$$

We assume that (A_i, B_i) is controllable for all i and we seek a controller K that stabilizes the collection of M systems. The following theorem shows one how to construct such a controller.

Theorem

A state-variable feedback controller $u = -Kx$ that stabilizes all of the systems in Eq. (1) exists if there exist Q_i and positive definite R_i such that

$$Q_i + P_i B_i R_i^{-1} B_i^t P_i + 2P_i B_i \left(\sum_{i \neq j} R_i^{-1} B_i^t P_i \right) > 0 \tag{2}$$

where P_i is the solution of the algebraic Riccati equation

$$A_i^t P_i + P_i A_i - P_i B_i R_i^{-1} B_i^t P_i + Q_i = 0 \tag{3}$$

Moreover, the controller that stabilizes the M systems is given by

$$K = -\sum_{i=1}^{M} R_i^{-1} B_i^t P_i \tag{4}$$

Remark: Note that Q_i need not be positive semidefinite as is usually assumed. It was shown in Ref. 13 that, in the standard